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# *Point Transformations in Elliptic Coordinates of Circles having Double Contact with a Conic.*

By E. O. Lovett.

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There are two systems of circles having double contact with a conic, the chords of contact of each system being parallel to one of the axes of the curve.

If

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is the conic, these two systems are

$$x^2 + y^2 - 2e^2x_1x + e^2x_1^2 - b^2 = 0, \qquad (1)$$

$$x^2 + y^2 + \frac{2c^2y_1}{b^2}y - \frac{c^2}{b^2}y_1^2 - a^2 = 0, \qquad (2)$$

corresponding respectively to the chords of contact

$$x - x_1 = 0, \quad y - y_1 = 0.$$

The equation (1) may be written in the form

$$x^2 + y^2 - 2cx\cos\theta + c^2\cos^2\theta - b^2\sin^2\theta = 0. \qquad (3)$$

This equation is transformed into its equivalent in elliptic coordinates

$$\mu^2 + \nu^2 - 2\mu\nu\cos\theta - a^2\sin^2\theta = 0, \qquad (4)$$

by assuming

$$\left.\begin{aligned} x^2 + y^2 &= \mu^2 + \nu^2 - c_2, \\ cx &= \mu\nu. \end{aligned}\right\} \qquad (5)$$

The equation (4) may be replaced by the relation

$$\cos^{-1}\frac{\mu}{a} \pm \cos^{-1}\frac{\nu}{a} = \theta, \qquad (6)$$

whence, by differentiation,

$$\frac{d\mu}{\sqrt{a^2 - \mu^2}} \pm \frac{d\nu}{\sqrt{a^2 - \nu^2}} = 0, \qquad (7)$$

the differential equation of the system of circles (1). Similarly we find that

$$\frac{\mu d\mu}{\sqrt{(\mu^2-a^2)(\mu^2-c^2)}} \pm \frac{\nu d\nu}{\sqrt{(a^2-\nu^2)(c^2-\nu^2)}} = 0 \tag{8}$$

is the differential equation of the second system of circles (2).

It is proposed now to find the general point transformations in the elliptic coordinates $\mu$, $\nu$ which leave these families of circles invariant.

Let the general infinitesimal transformation of the group which leaves the first system of circles invariant be

$$Uf \equiv \xi(\mu,\nu)\frac{\partial f}{\partial\mu} + \eta(\mu,\nu)\frac{\partial f}{\partial\nu}. \tag{9}$$

Putting $\dfrac{d\nu}{d\mu} \equiv \rho$, the general infinitesimal transformation of the first extension of the original group is

$$U'f \equiv \xi(\mu,\nu)\frac{\partial f}{\partial\mu} + \eta(\mu,\nu)\frac{\partial f}{\partial\nu} + \chi(\mu,\nu,\rho)\frac{\partial f}{\partial\rho}, \tag{10}$$

where

$$\chi(\mu,\nu,\rho) \equiv \eta_\mu + (\eta_\nu - \xi_\mu)\rho - \xi_\nu\rho^2. \tag{11}$$

By this transformation a function $\Omega(\mu,\nu,\rho)$ receives the increment

$$\delta\Omega \equiv U'\Omega\delta t \equiv \frac{\partial\Omega}{\partial\mu}\delta\mu + \frac{\partial\Omega}{\partial\nu}\delta\nu + \frac{\partial\Omega}{\partial\rho}\delta\rho, \tag{12}$$

where

$$\delta\mu = \xi(\mu,\nu)\delta t, \quad \delta\nu = \eta(\mu,\nu)\delta t, \quad \delta\rho = \chi(\mu,\nu,\rho)\delta t. \tag{13}$$

If the function $\Omega$ is to be invariant, then $\delta\Omega$ is zero.

The function $\Omega$ for the case in hand is given by the equation (7), which may be written

$$\Omega(\mu,\nu,\rho) \equiv \sqrt{a^2-\nu^2} \pm \rho\sqrt{a^2-\mu^2} = 0. \tag{14}$$

The total variation of $\Omega$ becomes

$$\delta\Omega \equiv \mu\rho^2\delta\mu - \nu\delta\nu \pm \sqrt{(a^2-\mu^2)(a^2-\nu^2)}\,\delta\rho = 0, \tag{15}$$

which by the relations (13) assumes the form

$$\{\mu\xi \mp \sqrt{(a^2-\mu^2)(a^2-\nu^2)}\}\rho^2 \pm (\eta_\nu - \xi_\mu)\rho - \nu\eta \pm \eta_\mu\sqrt{(a^2-\mu^2)(a^2-\nu^2)} = 0, \tag{16}$$

which must be true for all values of $\rho$; accordingly we find the following equations of condition for the functions $\xi(\mu,\nu)$ and $\eta(\mu,\nu)$:

$$\frac{\mu\xi}{\xi_\nu} = \frac{\nu\eta}{\eta_\mu} = \pm\sqrt{(a^2-\mu^2)(a^2-\nu^2)}, \tag{17}$$

$$\xi_\mu = \eta_\nu. \tag{18}$$

The logarithmic integration of equations (17) gives

$$\left.\begin{aligned} \xi &= e^{\pm \Phi(\mu) \sin^{-1} \frac{\nu}{a}}, \\ \eta &= e^{\pm \Psi(\nu) \sin^{-1} \frac{\mu}{a}}, \end{aligned}\right\} \tag{19}$$

in which

$$\Phi(\mu) = \frac{\mu \phi(\mu)}{\sqrt{a^2 - \mu^2}}, \quad \Psi(\nu) = \frac{\nu \psi(\nu)}{\sqrt{a^2 - \nu^2}}, \tag{20}$$

$\phi(\mu)$ and $\psi(\nu)$ being arbitrary functions. The equation (18) leads to the determination of the forms of $\Phi$ and $\Psi$, for, subjecting (19) to the condition (18), there results

$$\frac{\pm \Phi'(\mu) e^{\pm \Phi(\mu)}}{\sin^{-1} \frac{\mu}{a} e^{\sin^{-1} \frac{\mu}{a}}} \equiv \frac{\pm \Psi'(\nu) e^{\pm \Psi(\nu)}}{\sin^{-1} \frac{\nu}{a} e^{\sin^{-1} \frac{\nu}{a}}} = m\,; \tag{21}$$

whence, integrating by parts,

$$\Phi(\mu) = \pm \sin^{-1} \frac{\mu}{a} \pm \log \left\{ \frac{m}{2} \left[ (\mu + \sqrt{a^2 - \mu^2}) \sin^{-1} \frac{\mu}{a} - \mu \right] \right\}, \quad \Psi(\nu) = \Phi(\nu). \tag{22}$$

Accordingly the infinitesimal transformation sought has the form

$$Uf \equiv e^{\pm F(\mu, \nu)} \frac{\partial f}{\partial \mu} + e^{\pm F(\nu, \mu)} \frac{\partial f}{\partial \nu}, \tag{23}$$

where $\pm F(\mu, \nu) = \pm \Phi(\mu) \sin^{-1} \frac{\nu}{a}$.

In like manner we may find the infinitesimal point-transformations which leave invariant, respectively, the system of circles (8) and their orthogonal trajectories,

$$\frac{d\mu}{\mu} \sqrt{\frac{\mu^2 - a^2}{\mu^2 - c^2}} \mp \frac{d\nu}{\nu} \sqrt{\frac{a^2 - \nu^2}{c^2 - \nu^2}} = 0.$$

It is to be observed that we cannot pass from the form (23) to the form of $Uf$ in Cartesian coordinates by equations (5) solved for $\mu$ and $\nu$ in terms of $x$ and $y$, since the equations (5) do not represent a continuous group of transformations.

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